A FINITE-DIFFERENCE METHOD OF CALCULATING A TURBULENT BOUNDARY LAYER OF INCOMPRESSIBLE LIQUID

Yu. E. Karyakin and V. G. Sharov

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An implicit finite-difference method of solving the equations of a turbulent boundary layer of incompressible liquid in the presence or absence of injection at the wall is discussed. The results of calculations by this method are compared with experimental data.

The obtention of exact solutions of the equations of a turbulent boundary layer is a problem which has received little study so far. Recent progress in this area has been due to the application of numerical finite-difference methods of investigation [1-4], which allow a detailed calculation of turbulent boundary layers of the most diverse nature. The lack of information on the mechanism of turbulence, however, has led to a great variety of semiempirical hypotheses by different authors. The numerical methods of solving boundary-layer equations are also very diverse.

Below we describe a numerical method of calculating a turbulent boundary layer. This method gives results which are in good agreement with experimental data.

We consider the equations of averaged motion in a plane turbulent boundary layer of incompressible liquid

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\frac{1}{\rho}\frac{\partial\tau}{\partial y},\qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

where τ is the total tangential friction stress

$$\tau=\mu\frac{\partial u}{\partial y}-\rho\langle u'v'\rangle=(\mu+\varepsilon)\frac{\partial u}{\partial y}.$$

The system of equations (1) is solved with boundary conditions:

$$\text{for } y=0 \quad u=0;\quad v=v_w,\qquad \text{for } y\to\infty \quad u\to u_e, \tag{2}$$

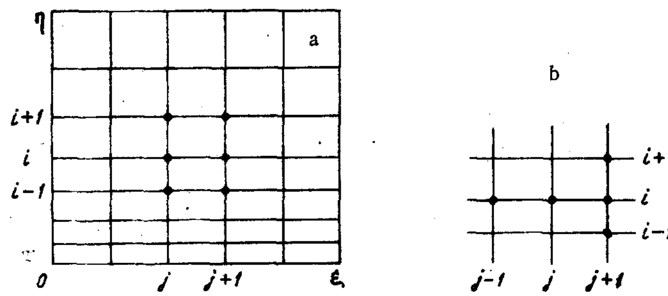

Fig. 1. Calculation grid and stencils used in approximation of differential equation (7) by a difference equation.

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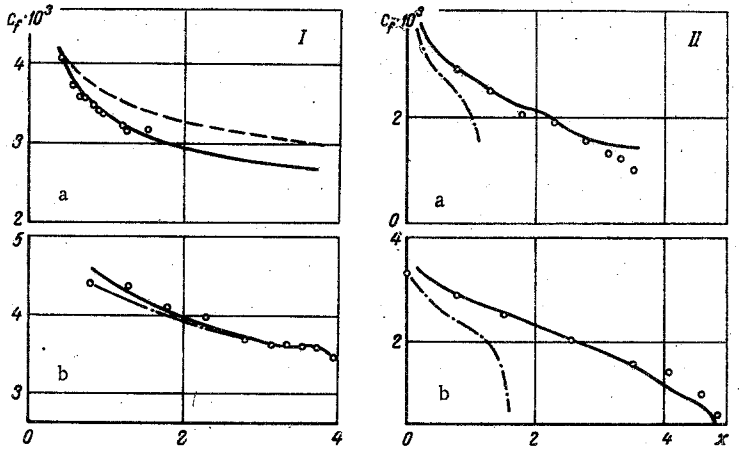

Fig. 2. Comparison of calculated and experimental values: I) local friction factor (a = flat plate, experiment 3000 [9]; b) negative pressure gradient, experiment 1300 [9]); II) local friction factor for positive pressure gradients (a = experiment 1200 [9]; b = experiment 4800 [9]). x, m.

where v_w is the velocity of suction or injection at the surface.

In the assignment of a relationship between the turbulent viscosity ε and the averaged characteristics of the motion we will divide the boundary layer into two regions: an inner one and an outer one. In the inner region, in the immediate vicinity of the wall, we will assume, in accordance with [5], that

$$\varepsilon = \varepsilon_i = \rho\,(0.4y)^2 \left[1 - \exp\left(-\frac{y}{a}\right)\right]^2 \cdot \left|\frac{\partial u}{\partial y}\right|, \tag{3}$$

where a is a damping constant; taking into account the pressure gradient and suction (injection) in the boundary layer we put this constant in the form [2]

$$a = \frac{26\nu}{v_*}\left\{-\frac{p^0}{v_w^0}\left[\exp(11.8v_w^0) - 1\right] + \exp(11.8v_w^0)\right\}^{-1/2}, \tag{4}$$

where

$$p^0 = -\frac{dp}{dx}\frac{\nu}{\rho v_*^3}, \quad v_w^0 = \frac{v_w}{v_*}, \quad v_* = \left(\frac{\tau_w}{\rho}\right)^{1/2}.$$

For the outer region of the boundary layer we use the alternation factor and put the turbulent viscosity in the form [2]

$$\varepsilon = \varepsilon_0 = 0.0168\rho \int_0^\infty (u_e - u)\,dy \left[1 + 5.5\left(\frac{y}{\delta_{0.995}}\right)^6\right]^{-1}. \tag{5}$$

The line of separation of the turbulent boundary layer into inner and outer regions is given by the condition

$$\varepsilon_i(x, y) = \varepsilon_0(x, y).$$

To facilitate the application of the numerical method of solution we use in equations (1) a Lees-type transformation of the variables [6]

$$\xi = \rho\mu \int_0^x u_e dx, \quad \eta = \frac{\rho u_e}{\sqrt{2\xi}}\,y, \quad f = \int_0^\eta \frac{u}{u_e}\,d\eta + f_w. \tag{6}$$

In these variables system (1) is written as ($F = \partial f/\partial\eta$):

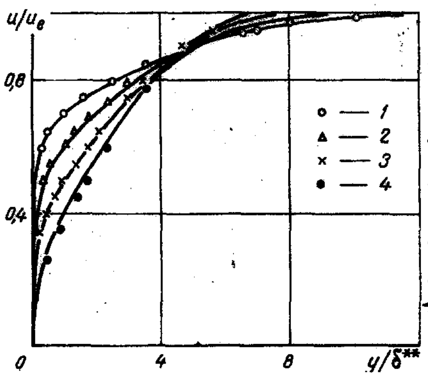

Fig. 3. Velocity profile u/u_e in boundary layer for different values of injection parameter v_w/u_e. Experimental data of [11]: 1) $v_w/u_e = 0$; 2) 0.0039; 3) 0.00993; 4) 0.0147.

$$\frac{\partial}{\partial\eta}\left(\alpha\frac{\partial F}{\partial\eta}\right)+f\frac{\partial F}{\partial\eta}+\beta\,(1-F^2)=2\xi\left(F\frac{\partial F}{\partial\xi}-\frac{\partial f}{\partial\xi}\frac{\partial F}{\partial\eta}\right) \tag{7}$$

with boundary conditions

$$\xi=\xi_0,\ F(\xi_0,\ \eta)=F_1(\xi_0,\ \eta);\ \eta=0,\ f=f_w,\ F=0;\ \eta\to\infty,\ F\to 1. \tag{8}$$

Here $\alpha=1+\frac{\varepsilon}{\mu}$, $\beta=2\xi\frac{1}{u_e}\frac{du_e}{d\xi}$.

To construct the numerical method of solving problem (7), (8) we replace the region of continuous variation of the arguments ξ and η by the calculation grid shown in Fig. 1.

Taking into account the particular nature of the change in quantities f and F, we introduce a variable step of the coordinate η across the boundary layer so that [3] the grid steps in the direction of this coordinate form an increasing geometric progression with denominator k:

$$\eta_{i+1}=\eta_i+h_i;\ h_i=kh_{i-1},\ i=0,\ 1,\ \ldots,\ N-1,$$

the step h_0 close to the wall has to be assigned.

Then, for any i-th layer of the grid region

$$\eta_i=h_0\frac{k^i-1}{k-1},\quad i=0,\ 1,\ \ldots,\ N;\ k\neq 1.$$

On any layer ξ = const of the grid region we use the following expressions for the approximation of the differential operators F' and $(\alpha F')'$ contained in (7) by difference operators (the dash means the derivative with respect to η):

$$F_i'=\frac{F_{i+1}-F_{i-1}}{\eta_{i+1}-\eta_{i-1}}, \tag{9}$$

$$(\alpha F')_i'=\frac{2\left[\alpha_{i+1/2}(F_{i+1}-F_i)(\eta_i-\eta_{i-1})-\alpha_{i-1/2}(F_i-F_{i-1})(\eta_{i+1}-\eta_i)\right]}{(\eta_{i+1}-\eta_{i-1})(\eta_{i+1}-\eta_i)(\eta_i-\eta_{i-1})}. \tag{10}$$

The derivatives $\partial F/\partial\xi$ and $\partial f/\partial\xi$ contained in (7) were approximated by two different methods with the aim of selecting the better method. Accordingly, in the construction of the difference analog of problem (7) we considered two different stencils (Fig. 1a and b), i.e., groups of grid points adjacent to the fixed point at which the values of the grid functions F and f are used as approximations for the operators of the differential equation.

a) For a six-point stencil (Fig. 1a) we use a weighted scheme [7]. The difference analogs of the quantities contained in (7) are put in the form

$$\psi=\sigma\psi_{i,j+1}+(1-\sigma)\psi_{i,j},$$
$$\frac{\partial\Phi}{\partial\xi}=\frac{\Phi_{i,j+1}-\Phi_{i,j}}{\xi_{j+1}-\xi_j}, \tag{11}$$

where ψ is any of the quantities $(\alpha F')'$, f, F', β, F^2, ξF, ξ: Φ is any of the grid functions F or f, and σ is a real parameter called the weight of the scheme. When $\sigma = 0$ we obtain a four-point explicit scheme, and when $\sigma = 1$ we have an advance scheme or a purely implicit scheme.

Using (11), expressions (9) and (10), and expressions similar to them, we can write for the differential equation (7) the corresponding difference equation, which after elementary algebra can be brought to the form

$$a_i F_{i-1,j+1} + b_i F_{i,j+1} + c_i F_{i+1,j+1} = g_i, \quad i = 1, 2, \ldots, N-1. \tag{12}$$

Coefficients a_i, b_i, c_i, and g_i are variables which depend on the values of the required functions f and F.

b) If a five-point stencil is used (Fig. 1b) the derivatives $F'_{i,j+1}$ and $(\alpha F')'_{i,j+1}$ are written in form (9) and (10) respectively, and the derivatives with respect to coordinate ξ are represented with the aid of the left-hand three-point scheme as:

$$\left.\frac{\partial \Phi}{\partial \xi}\right|_{i,j+1} = \frac{3\Phi_{i,j+1} - 4\Phi_{i,j} + \Phi_{i,j-1}}{\xi_{j+1} - \xi_{j-1}}. \tag{13}$$

By Φ here we mean any of the grid functions F or f. Relation (13) can be used in the whole region, of course, with the exception of the case j = 0. In this case the derivative is approximated by a two-point scheme.

As a result of approximation of the differential operators by means of these relations in the case of a five-point stencil the differential equation (7) can be replaced by a difference equation in a form similar to (12).

For any of the stencils we obtain the solution of the difference equation (12) by iteration, using the pivot method on each layer ξ = const [7].

The boundary value f_w on the wall (if there is no injection, $f_w \equiv 0$) will satisfy the equation

$$\frac{df_w}{d\xi} + \frac{f_w}{2\xi} = -\frac{v_w}{\mu u_e \sqrt{2\xi}}, \tag{14}$$

which is easily integrated and gives

$$f_w = -\frac{1}{\mu\sqrt{2\xi}} \int_0^{\xi} \frac{v_w}{u_e}\, d\xi. \tag{15}$$

In the case where a six-point weighted scheme is used (Fig. 1a), however, the direct application of (15) can lead to errors in the assignment of the difference boundary condition on the wall. Hence, for such a pattern the operators of equation (14) are approximated by difference operators in accordance with the weighted scheme. As a result, on each new layer f_{wj+1} is found from the relation

$$f_{wj+1} = f_{wj}\left[1 - \frac{1}{3\sigma + \xi^0}\right] - \frac{\sqrt{2}}{\mu}\, \frac{[\sigma\xi_{j+1} + (1-\sigma)\,\xi_j]^{1/2}}{3\sigma + \xi^0} \left[\sigma\, \frac{v_{wj+1}}{u_{ej+1}} + (1-\sigma)\, \frac{v_{wj}}{u_{ej}}\right], \quad \xi^0 = \frac{2\xi_j}{\xi_{j+1} - \xi_j}.$$

To solve equation (12) by the numerical method we compiled a program for the M-220 computer.

The mass of values for the outer flow velocity, compiled from experimental data, were first smoothed by a standard procedure involving a spline function [8]. The application of this procedure also allowed a very accurate calculation of the derivatives of the external velocity with respect to the longitudinal coordinate.

The calculation procedure begins at cross section x = 0, where equation (7) degenerates into an ordinary differential equation. Since a real developed turbulent boundary layer begins not at point x = 0, but at some transition point $x = x_t$, we used the following method of calculation: for $x < x_t$ the boundary layer was assumed to be laminar with $\alpha = 1$; for $x > x_t$ the boundary layer was regarded as a developed turbulent layer and the values of α were determined from (3)-(5). As the calculation results showed, a change in x_t for flows with zero or negative pressure gradients mainly affects the characteristics of the turbulent boundary layer in sections situated in the immediate vicinity of the point $x = x_t$. In flows with positive pressure gradients the variation of x_t leads to a parallel shift of the integral characteristics of the turbulent boundary layer. In this case, as the calculations showed, the value of x_t has to be chosen so that any characteristic of the turbulent boundary layer, e.g., $c_f = c_f(x)$, passes through a fixed point in its own plane; in our example this is the point (c_{f0}, x_0), which is determined from the results of an appropriate experiment or by some other method.

It goes without saying that the assignment of a velocity profile at some section x_0 of the turbulent boundary layer entirely obviates the search for the value of x_t.

Using the expounded method we carried out systematic calculations of the turbulent boundary layer. The obtained results were compared with the experimental data given in [9]. In addition, we attempted to estimate the error introduced into the calculations by the hypothesis of local similarity, which is widely used in the theory of a turbulent boundary layer [10]. For this purpose for the majority of considered cases of flow we integrated the boundary layer equation (7) also in its local approximation, which is obtained by discarding the derivatives of the required functions with respect to coordinate ξ on the right hand side of (7).

For the majority of calculations we put $h_0 = 0.005$ and $k = 1.07$ with the number of points across the boundary layer equal to 100. A significant increase in the number of these points did not lead to any appreciable changes in the results. The weight σ of the difference scheme was chosen in the range 0.5-1.0. The calculations showed that the use of a six-point stencil (scheme a) and a five-point stencil (scheme b) led to practically the same results, which were in good agreement with the experimental data in the main region of flow.

Figure 2 gives some results of numerical calculations of a turbulent boundary layer in an incompressible liquid in the absence of injection at the surface. The solid curves are the results obtained by the described finite-difference method. The points are the experimental data.

Figure 2, Ia shows the values of the local friction factor of a flat plate. The dashed line on the same figure gives the values of this factor calculated from the formula

$$c_f = 0.0263\mathrm{Re}_x^{-1.7}, \tag{16}$$

which is often used for calculation of a flat plate. Figure 2, Ib gives the values of c_f for the case of a negative pressure gradient, while Fig. 2, IIa, b gives the values for the case of positive pressure gradients. The agreement of the results of the finite-difference method with the experimental data in these cases can be regarded as good.

The dot-dash lines in Fig. 2, Ib and 2, IIa, b are the curves of the local friction factor calculated by using the above-mentioned hypothesis of local similarity. It is obvious that the hypothesis gives acceptable results only in the case of flows with negative pressure gradients.

Figure 3 gives the results of calculation of a turbulent boundary layer on a plate in the presence of injection of substance into the boundary layer. The points represent the experimental data of [11]. We can conclude from a comparison of the obtained curves with the experimental data that the method satisfactorily predicts the distortion of the velocity profile even at fairly high values of the injection parameter.

NOTATION

x, y	are the coordinates directed along the normal to surface of body;
u, v	are the longitudinal and transverse velocity components;
p, ρ, μ, ν	are the pressure, density, dynamic viscosity, and kinematic viscosity of liquid;
τ	is the total tangential friction stress;
c_f	is the local friction factor;
ε	is the dynamic turbulent viscosity (ε_i, ε_0, in inner and outer regions of boundary layer, respectively);
a	is the damping constant;
$v_* = (\tau_w/p)^{1/2}$	is the dynamic velocity;
$\delta_{0.995}$, δ^{**}	are the value of coordinate y used for thickness of boundary layer for $F = 0.995$ and momentum loss thickness;
ξ, η	are the Lees variables;
f, F	are the new dependent variables;
h_i	is the grid step;
k	is the denominator of step;
N	is the number of grid points across boundary layer;
σ	is the weight of difference scheme;
a_i, b_i, c_i, g_i	are the coefficients of difference equation;
Φ	is a grid function;

$Re_x = u_e x/\nu$ is the Reynolds number;
$\alpha = 1 + \varepsilon/\mu$;
$\beta = 2\xi\ (1/u_e)\ (du_e/d\xi)$;
$p^0 = -(dp/dx)\ (\nu/\rho v_*^3)$;
$v_w^0 = v_w/v_*$.

Subscripts

e is the outer boundary of boundary layer;
w is the body surface;
t is the point of transition of laminar boundary layer to turbulent one.

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